

# A mathematical framework for reducing the domain in the mechanical analysis of periodic structures

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## Abstract

A theoretical framework is developed leading to a sound derivation of Periodic Boundary Conditions (PBCs) for the analysis of domains smaller than the Unit Cells (UCs), named reduced Unit Cells (rUCs), by exploiting non-orthogonal translations and symmetries. A particular type of UCs, Offset-reduced Unit Cells (OrUCs) are highlighted. These enable the reduction of the analysis domain of the traditionally defined UCs without any loading restriction. The relevance of the framework and its application to any periodic structure is illustrated through two practical examples: 3D woven and honeycomb.

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## 1. Introduction

Numerical analysis of periodic materials and structures has proven to be an extremely powerful tool. It provides detailed information, such as failure initiation sites and stress-strain at smaller scales (meso/micro). It has been successfully used to determine homogenised properties, study the detailed stress-strain fields at nano- and microscopic scales to obtain structural damage initiation conditions and sites, as well as to simulate damage development and associated deterioration of the homogenised mechanical properties [1]. Several works can be found discussing the application of periodic boundary conditions to representative regions, e.g. [2–5]. For periodic structures, the Unit Cell (UC) is used as the representative region, and the analysis is performed by applying periodic displacement boundary conditions. The topological complexity of many UCs found in practice, such as in typical woven composites, often leads to unpractical modelling and analysis times. For this reason, internal symmetries of the UCs must whenever possible be exploited to reduce the analysis

domain further (provided the appropriate boundary conditions are applied), thus reducing both modelling and analysis time.

A comprehensive study on the determination of reduced Unit Cells (rUCs) for UD and particle reinforced composites was performed by Li [6, 7] and Li and Wongsto [8]. Different rUCs, loading cases and correspondent boundary conditions were determined and presented in detail. Applied to textile composites, Tang and Whitcomb [9] proposed a general framework for determining rUCs.

In the first part of the present work, the derivation of the framework proposed in [9] is revisited and some of its building blocks redefined, resulting in a different, formally defined and more concise formulation. The framework proposed by Tang and Whitcomb [9] requires the distinction of two different cases of equivalence between subcells: (i) equivalence is obtained by a symmetry operation or a translation, and (ii) equivalence is obtained by the combination of a symmetry operation and a translation. In the second case an additional vector of constants  $\mathbf{r}$  (see Tang and Whitcomb [9]) needs to be considered when applying the boundary conditions. The non-zero components of this vector are tabulated for different cases and are determined by the FEM as part of the solution. The formulation derived in the present work is more generic, in that no cases need to be treated separately, and mathematically complete in that no vector  $\mathbf{r}$  needs to be determined from tabulated data. All terms in the equation that assigns the periodic boundary conditions for a rUC are fully defined, simplifying the formulation and consequently their use.

In the second part of this paper, the application of the formulation developed and its potential is illustrated through two practical examples: 3D woven composites and honeycombs. Additionally, particular attention is given to Offset-reduced Unit Cells as they allow the domain reduction without load restrictions.

## **2. Equivalence framework**

In this section, the equivalence framework is formally defined. It is based on four concepts: physical equivalence, load equivalence, periodicity and load admissibility. In the following sections each of these concepts is detailed.

### 2.1. Physical equivalence

Consider a domain  $\mathbf{D}$  in space and within it a sub-domain  $E$ . The latter has a defined boundary, Local Coordinate System (LCS)  $O_Exyz$ , and a certain spatial distribution of  $n$  physical properties  $\mathbf{P}^i$  with  $i \in \{1, \dots, n\}$ . Each of these physical properties  $\mathbf{P}^i$  are expressed as a tensor written in the LCS of  $E$ , i.e.  $\mathbf{P}_E^i$ .

**Definition 1.** Two distinct sub-domains  $E$  and  $\hat{E}$  are Physically Equivalent, denoted:

$$E \cong \hat{E} \quad (1)$$

if for every point  $A$  in  $E$  there is a point  $\hat{A}$  in  $\hat{E}$  such that, for each physical property  $i$ ,

$$\mathbf{x}_E^A = \mathbf{x}_{\hat{E}}^{\hat{A}} \wedge \mathbf{P}_E^i(A) = \mathbf{P}_{\hat{E}}^i(\hat{A}) \quad (2)$$

and vice-versa.

In Eq. 2,  $\mathbf{x}_E^A$  and  $\mathbf{x}_{\hat{E}}^{\hat{A}}$  are the coordinate vectors of  $A$  and  $\hat{A}$  given in the LCS  $O_Exyz$  and  $O_{\hat{E}}xyz$  associated with  $E$  and  $\hat{E}$ , respectively, Fig. 1. The points  $A$  and  $\hat{A}$  for which Eq. 1 is verified are designated as physically equivalent points.

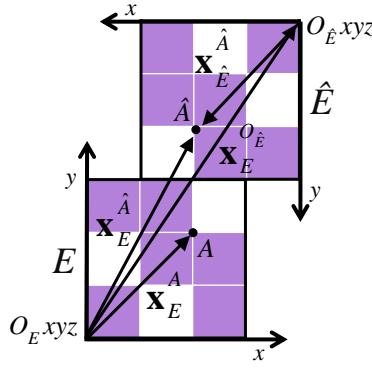


Figure 1: Geometrical relation between equivalent points.

### 2.2. Periodicity and Unit Cell

Across the literature, different definitions can be found for periodic structure and UC. In the present work, periodic structure and UC are defined based on the concept of physical equivalence.

**Definition 2.** A domain  $D$  is periodic if it can be reconstructed by tessellation of, non-overlapping, physically equivalent sub-domains  $E_i$  with parallel LCS, i.e. if for all  $i \neq j$ :

$$E_i \hat{=} E_j \wedge O_{E_i}xyz \parallel O_{E_j}xyz \quad (3)$$

The smallest sub-domain verifying the periodicity definition is designated as an Unit Cell.

### 2.3. Loading equivalence

The concept of load equivalence (see Tang and Whitcomb [9]) provides a relation between physically-equivalent sub-domains, once the structure they are part of is loaded. Let us consider a periodic structure as defined in the previous section.

**Definition 3.** Load equivalence between two physically equivalent points  $A$  and  $\hat{A}$  is verified if the strains and stresses at these points, given in the LCS of the sub-domains, can be related by:

$$\varepsilon_E(A) = \gamma \varepsilon_{\hat{E}}(\hat{A}) \quad (4)$$

$$\sigma_E(A) = \gamma \sigma_{\hat{E}}(\hat{A}), \quad (5)$$

where the load reversal factor,  $\gamma = \pm 1$ , is used to enforce the equivalence between fields of physically equivalent sub-domains.

For Eqs. 4 and 5 to hold, the length scale of the loading variation must be larger than the length scale of the sub-domains, such that an approximately periodic variation of the strains and stress fields is assured. If a structure is entirely composed by load equivalent sub-domains, its response can be obtained by analysing one of these domains alone, instead of analysing the entire structure. However, the appropriate boundary conditions have to be applied. These guarantee that the sub-domain, although isolated, has the same response it would have if it was embedded in the structure.

### 2.4. Sub-domain admissibility

Not all physically equivalent sub-domains can be used to analyse the response of a periodic structure under all loading conditions. The use of sub-domains smaller than the UC to analyse the response of a periodic structure is restricted by the relations between the LCS of these sub-domains. The sufficient and necessary condition for admissibility of a sub-domain to be used in the analysis of a periodic structure is derived below.

### Average and fluctuation fields

For convenience, the strain field of a sub-domain at a point  $A$  is decomposed as the sum of a volume average and a fluctuation term, see Fig.2:

$$\varepsilon(A) = \langle \varepsilon \rangle + \varepsilon^*(A), \quad (6)$$

where  $\langle \bullet \rangle = \frac{1}{V} \int_V \bullet dV$  is the volume average operator over the volume  $V$ , and  $\varepsilon^*$  is the fluctuation term, see Suquet [10]. It is possible to find the displacements at a given point by integration of Eq. 6. Assuming small displacements and no rigid body rotations, the displacement relative to the origin of a LCS, attached to the subdomain, comes as:

$$\mathbf{u}(A) = \langle \varepsilon \rangle \cdot \mathbf{x}^A + \mathbf{u}^*(A). \quad (7)$$

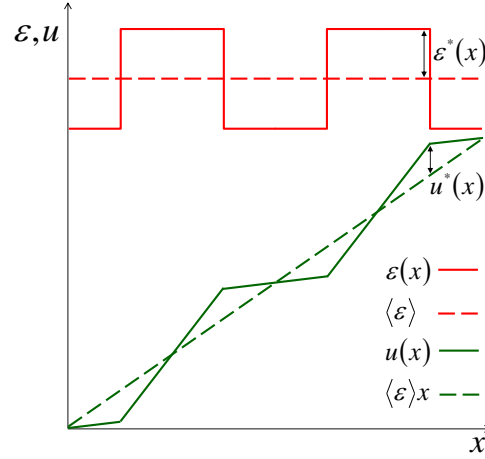


Figure 2: Idealised relation between the fluctuation and average fields of strain and displacement in a periodic structure.

### Relations between fields of two equivalent points in a global coordinate system

Knowing that the coordinate vectors of two equivalent points  $A$  and  $\hat{A}$  given in their LCSs are identical, Eq. 2, they can be related in the LCS of the sub-domain  $E$  by:

$$\mathbf{x}_E^A = \mathbf{T} \left( \mathbf{x}_E^{\hat{A}} - \mathbf{x}_E^{O_{\hat{E}}} \right) \quad (8)$$

where  $\mathbf{T}$  is the transformation matrix between the LCSs of  $\hat{E}$  and  $E$ , and  $\mathbf{x}_E^{O_{\hat{E}}}$  is the position vector of the origin of the LCS of the sub-domain  $\hat{E}$  given in the LCS of the sub-domain  $E$ , Fig. 1.

Similarly, using Eq. 4, the strains at two equivalent points can be related in the LCS of  $E$  by:

$$\varepsilon_E(A) = \gamma \mathbf{T} \varepsilon_E(\hat{A}) \mathbf{T}^t. \quad (9)$$

The relation between the volume average of the strain of the equivalent sub-domains  $E$  and  $\hat{E}$ , in the LCS of the first, can be obtained directly by integrating Eq. 9:

$$\langle \varepsilon \rangle_E^E = \gamma \mathbf{T} \langle \varepsilon \rangle_E^{\hat{E}} \mathbf{T}^t, \quad (10)$$

where the lower index of  $\langle \bullet \rangle$  refers to the coordinate system, and the upper index to the domain over which the volume average was taken. Decomposing the strain field in Eq. 9 into its average and fluctuation parts and using Eq. 10, the relation between the strain fluctuations field of two equivalent points is obtained:

$$\varepsilon_E^*(A) = \gamma \mathbf{T} \varepsilon_E^*(\hat{A}) \mathbf{T}^t. \quad (11)$$

In general, the displacement of a point  $A$  can be obtained from:

$$\mathbf{u}(\mathbf{x}^A) = \mathbf{u}_0 + \int_{\mathbf{x}=0}^{\mathbf{x}^A} d\mathbf{u} = \mathbf{u}_0 + \int_{\mathbf{x}=0}^{\mathbf{x}^A} \nabla \cdot \mathbf{u} d\mathbf{x} = \mathbf{u}_0 + \int_{\mathbf{x}=0}^{\mathbf{x}^A} \varepsilon d\mathbf{x} + \int_{\mathbf{x}=0}^{\mathbf{x}^A} \Omega d\mathbf{x}, \quad (12)$$

where  $\varepsilon = \frac{1}{2} (\nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{u}^t)$ ,  $\Omega = \frac{1}{2} (\nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{u}^t)$  and  $\mathbf{x}^A$  is the coordinate vector of point  $A$ .

Considering the structure has no rigid body motion  $\mathbf{u}_0 = 0$ , nor rotation  $\Omega = 0$ , the displacement of a point  $A$  can be simply obtained by:

$$\mathbf{u}(\mathbf{x}^A) = \int_{\mathbf{x}=0}^{\mathbf{x}^A} \varepsilon d\mathbf{x}. \quad (13)$$

The strain fluctuations of two equivalent points belonging to the sub-domains  $E$  and  $\hat{E}$  are related in the LCS of  $E$  by (see also Eq. 11):

$$\varepsilon_E^*(\mathbf{x}_E^A) = \gamma \mathbf{T} \varepsilon_E^*(\mathbf{x}_E^{\hat{A}}) \mathbf{T}^t. \quad (14)$$

Knowing that two equivalent points are related by (see also Eq. 8):

$$\mathbf{x}_E^A = \mathbf{T} \left( \mathbf{x}_E^{\hat{A}} - \mathbf{x}_E^{O_{\hat{E}}} \right),$$

and that the displacement at the origin is equal to zero, integrating Eq. 14 it is possible to obtain:

$$\begin{aligned} \int_{\mathbf{x}_E=0}^{\mathbf{x}_E^A} \varepsilon_E^{E*}(\mathbf{x}) d\mathbf{x} &= \int_{\mathbf{x}_E=0}^{\mathbf{x}_E^A} \gamma \mathbf{T} \varepsilon_E^{\hat{E}*} \left( \mathbf{T}^t \mathbf{x}_E + \mathbf{x}_E^{O_{\hat{E}}} \right) \mathbf{T}^t d\mathbf{x}_E \\ \mathbf{u}_E^*(\mathbf{x}_E^A) &= \gamma \mathbf{T} \left\{ \mathbf{u}_E^*(\mathbf{T}^t \mathbf{x}_E^A + \mathbf{x}_E^{O_{\hat{E}}}) - \mathbf{u}_E^*(\mathbf{x}_E^{O_{\hat{E}}}) \right\}. \end{aligned} \quad (15)$$

Equation 15 provides a relation between the displacement perturbations of two equivalent points given in the LCS of one of the sub-domains. Apart from  $\mathbf{u}^*(\mathbf{x}_E^{O_{\hat{E}}})$ , all variables are known; below it is shown that  $\mathbf{u}^*(\mathbf{x}_E^{O_{\hat{E}}}) = 0$ .

According to the definition of periodicity, a periodic structure can be reconstructed from physically equivalent sub-domains with parallel coordinate systems. The strain fields at two equivalent points belonging to different sub-domains are related by:

$$\varepsilon_E(\mathbf{x}_E^A) = \gamma \mathbf{T} \varepsilon_E \left( \mathbf{T}^t \mathbf{x}_E^A + \mathbf{x}_E^{O_{\hat{E}}} \right) \mathbf{T}^t. \quad (16)$$

If we consider that the sub-domains are UCs, since the coordinate systems are parallel, the matrix  $\mathbf{T}$  will be equal to the identity matrix. Moreover, all equivalent sub-domains will be admissible and have a load reversal factor  $\gamma = 1$ . Equation 16 can then be simply written as:

$$\varepsilon_E(\mathbf{x}_E^A) = \varepsilon_E \left( \mathbf{x}_E^A + \mathbf{x}_E^{O_{\hat{E}}} \right). \quad (17)$$

If the two UCs being considered are adjacent, a vector  $\mathbf{d} = \mathbf{x}_E^{O_{\hat{E}}}$  can be defined and Eq. 17 can be generalized for any point  $\mathbf{x}$  of the structure:

$$\varepsilon(\mathbf{x}) = \varepsilon(\mathbf{x} + \mathbf{d}), \quad (18)$$

where  $\mathbf{d}$  is commonly named the periodicity vector, and corresponds to the period of the function  $\varepsilon(\mathbf{x})$  [10].

The integral of a periodic function  $f$  of period  $D$  can always be written as:

$$\int f(t) dt = g(t) + \bar{f}t + C, \quad (19)$$

where  $g(t)$  is also a periodic function of period  $D$ ,  $\bar{f}$  is the average the periodic function  $f$ , and  $C$  is a constant [11]. Using Eq. 19 it is possible to write:

$$\mathbf{u}^*(\mathbf{x}) = \int \varepsilon^*(\mathbf{x}) d\mathbf{x} = \mathbf{h}(\mathbf{x}). \quad (20)$$

The average term that would appear in Eq. 20 is zero since by definition  $\varepsilon^*(\mathbf{x})$  has zero average. Additionally, knowing that at the origin  $\mathbf{u}^*(\mathbf{x})$  is equal to zero, one can conclude that  $C$  will also be zero and thus  $\mathbf{u}^*(\mathbf{x})$  will be a periodic function with zero average. Using the above result, and knowing that the integration over a period of a periodic function with zero average is equal to zero, one can integrate both sides of Eq. 15 over a period:

$$\begin{aligned} \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{d}} \mathbf{u}^*(\mathbf{x}) d\mathbf{x} &= \gamma \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{d}} \mathbf{T} \left\{ \mathbf{u}^*(\mathbf{T}^t \mathbf{x} + \mathbf{x}_E^{O_{\hat{E}}}) - \mathbf{u}^*(\mathbf{x}_E^{O_{\hat{E}}}) \right\} d\mathbf{x} \\ 0 &= \gamma \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{d}} \mathbf{u}^*(\mathbf{x}_E^{O_{\hat{E}}}) d\mathbf{x}, \end{aligned} \quad (21)$$

obtaining:

$$\mathbf{u}^*(\mathbf{x}_E^{O_{\hat{E}}}) = 0. \quad (22)$$

Substituting Eq. 22 in Eq. 15, the relation between the displacement perturbations at two equivalent points can be finally obtained:

$$\mathbf{u}_E^*(\mathbf{x}_E^A) = \gamma \mathbf{T} \mathbf{u}_E^*(\mathbf{x}_E^{\hat{A}}). \quad (23)$$

#### *Evaluation of the sub-domain admissibility*

For a sub-domain to be admissible, the volume average (homogenised) strain calculated for this sub-domain on a given reference system must equal that volume average on any other sub-domain (on the same reference system), as the volume average is a homogenised entity, hence independent of the sub-domain where it was calculated. From load equivalence, the strains at physically-equivalent points are related (Eq. 5). Eq. 11 is obtained by simply integrating this relation over the sub-domain, but does not enforce directly that the volume average strain is a macroscopic entity independent of the



particular sub-domain. For the sub-domain to be admissible, the following condition must be verified:

$$\langle \varepsilon \rangle_E^E = \langle \varepsilon \rangle_E^{\hat{E}} \quad (24)$$

as, for a sub-domain to be admissible, the homogenised strain on a given reference system (in this case  $E$ ) must be the same for any sub-domain (in this case  $E$  and  $\hat{E}$ ). Therefore, Eq. 10 with Eq. 24 lead to the condition of sub-domain admissibility, as defined below.

**Definition 4.** *A given sub-domain  $E$  is admissible for the analysis of a periodic structure under a given loading  $\langle \varepsilon \rangle_E$ , if  $\mathbf{T}_i$  and  $\gamma_i$  correspondent to any other sub-domain  $\hat{E}_i$  are such that, for all  $\hat{E}_i$ :*

$$\langle \varepsilon \rangle_E^E = \gamma_i \mathbf{T}_i \langle \varepsilon \rangle_E^{\hat{E}_i} \mathbf{T}_i^t \quad (25)$$

Equation 25 can be used to, for a given loading, determine the load reversal factors  $\gamma_i$  associated with each of the sub-domains. The admissibility of a subdomain for structural analysis leads to the definition of a rUC.

**Definition 5.** *A reduced Unit Cell is a domain, smaller than the Unit Cell, that can be used to determine the response of a periodic structure to a given loading. The condition to be verified by a reduced Unit Cell in structural analysis is defined by Eq. 25.*

### 3. Derivation of Periodic Boundary conditions

To ensure the response of a periodic structure under a given loading can be determined from the response of a rUC, the appropriate boundary conditions that must applied to the latter need to be determined. In this section, the equivalence framework, presented previously, is used to derive the periodic boundary conditions for the analysis of rUCs.

Consider two adjacent sub-domains  $E$  and  $\hat{E}$  that are physically and load equivalent. If a point  $\hat{A}$  belonging to  $\hat{E}$  is chosen to be at the boundary of the sub-domain  $E$ , then its equivalent point  $A$  is also be at the boundary of  $E$ . Since both points  $A$  and  $\hat{A}$  belong to  $E$ , the displacement at each point can be obtained using Eq. 7:

$$\mathbf{u}(A) = \langle \varepsilon \rangle \mathbf{x}^A + \mathbf{u}^*(A) \quad (26)$$

$$\mathbf{u}(\hat{A}) = \langle \varepsilon \rangle \mathbf{x}^{\hat{A}} + \mathbf{u}^*(\hat{A}) \quad (27)$$

All quantities in Eqs. 26 and 27 are written in the LCS of  $E$ , and the volume average is taken over the sub-domain  $E$  (the subscript will be omitted hereafter for convenience). Since both points are equivalent, their positions are related by Eq. 8 leading to:

$$\mathbf{u}(A) = \langle \varepsilon \rangle \mathbf{T} (\mathbf{x}^{\hat{A}} - \mathbf{x}^{O_{\hat{E}}}) + \mathbf{u}^*(A) \quad (28)$$

Knowing that the displacement fluctuations at two equivalent points are related by Eq. 23, if Eq. 27 is multiplied by  $\gamma \mathbf{T}$  and then subtracted to Eq. 26, the displacement fluctuations cancel, leading to:

$$\mathbf{u}(A) - \gamma \mathbf{T} \mathbf{u}(\hat{A}) = (\langle \varepsilon \rangle \mathbf{T} - \gamma \mathbf{T} \langle \varepsilon \rangle) \mathbf{x}^{\hat{A}} - \langle \varepsilon \rangle \mathbf{T} \mathbf{x}^{O_{\hat{E}}} \quad (29)$$

Provided the sub-domain  $\hat{E}$  is admissible, see Definition 4, the term  $(\langle \varepsilon \rangle \mathbf{T} - \gamma \mathbf{T} \langle \varepsilon \rangle)$  is zero. Using this result, Eq. 29 can be simplified to Eq. 30, which is the main outcome of this analysis and can be used directly to apply periodic boundary conditions to a sub-domain:

$$\mathbf{u}(A) - \gamma \mathbf{T} \mathbf{u}(\hat{A}) = -\langle \varepsilon \rangle \mathbf{T} \mathbf{x}^{O_{\hat{E}}} \quad (30)$$

Once a displacement constraint equation is associated to all points at the boundary of the sub-domain  $E$ , loading can be applied by prescribing a volume average strain  $\langle \varepsilon \rangle$ .

It is relevant to notice that the displacement constraint equation traditionally used to impose periodic boundary conditions on a UC, see Suquet [10] for example, is a particular case of Eq. 30 where the matrix  $\mathbf{T}$  is equal to the identity matrix  $\mathbf{I}$ , since the LCSs of the UCs are parallel by definition and consequently, from the sub-domain admissibility evaluation, the load reversal factor is equal to one. It is important to highlight the differences between the result obtained above and the one obtained in Tang and Whitcomb [9]; as referred before, Eq. 30 is completely generic and self sufficient: no distinction needs to be made, in the current formulation, between the type of operation needed to achieve equivalence between subdomains. Moreover, all terms in Eq. 30 are fully defined, and can therefore be readily used to prescribe periodic boundary conditions to a given subdomain.

### 3.1. Offset reduced Unit Cells

According to the periodicity definition given in 2.2, a UC is the smallest sub-domain that allows a periodic structure to be reconstructed by tessellation of sub-domains that are physically equivalent to

the UC and have parallel LCS. Nevertheless, in most applications the UC is defined such that the LCS are not only parallel but orthogonally translated, Fig. 3a. However, smaller UCs can in general be defined if non-orthogonal translations are considered, Fig. 3b. Although, according to the definition, the representative sub-domains obtained through non-orthogonal translation are in fact UCs, in the present paper they are referred to as Offset-reduced Unit Cells, since they lead to a reduction in the domain of the traditionally defined UCs, Fig. 3.

An important feature of OrUCs is that all loading combinations are admissible. This key feature has been identified by Li [7] and used in the derivation of PBCs for rUCs of UD composites, and cracked laminates [12]. It is relevant to highlight that using the present formulation this feature comes as a natural result: since the LCS of all sub-domains are parallel, they relate to each other by the identity matrix, i.e.  $\mathbf{T} = \mathbf{I}$ , as a consequence Eq. 25 is always verified and therefore all loading cases are admissible.

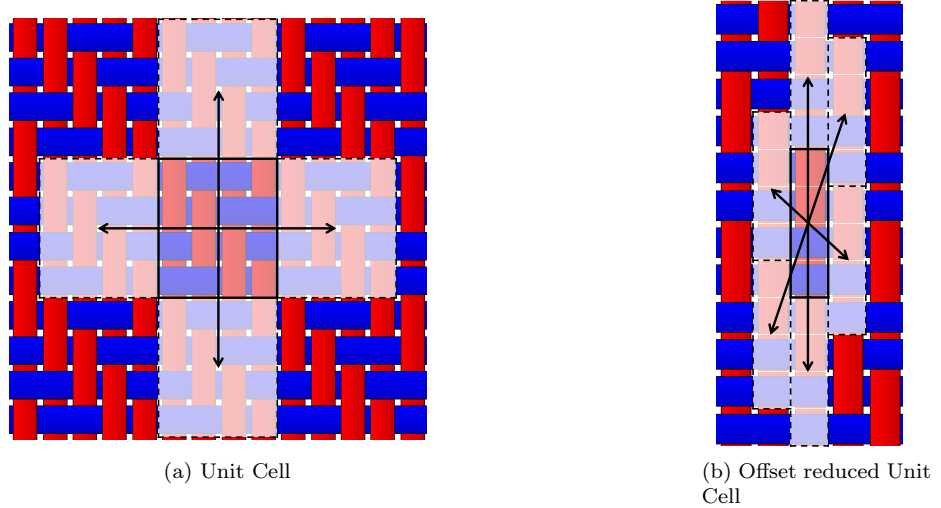


Figure 3: a) Unit Cell (UC) and b) Offset reduced Unit Cell (OrUC) of a  $2 \times 2$  Twill weave

#### 4. Applications

In the present section two applications of the formulation presented previously are illustrated.

#### 4.1. 3D Woven Composites

The UCs of 3D woven composites can be significantly larger than their 2D counterparts, mostly due to the more intricate reinforcement architecture and 3D nature. Therefore, the domain reduction enabled by the use of rUCs can be very significant. Figure 4, shows an UC, an OrUC and a rUC of a given 3D woven architecture, highlighting the domain reduction achieved: OrUC and rUC reduce the analysis domain to  $1/7$  and  $1/28$  of the UC, respectively.

To define the periodic boundary conditions for the analysis of the rUC of Fig. 4, Eq. 30, the geometric relations between equivalent points at the rUC boundary need to be found. These are obtained by applying Eq. 8 to the equivalent domains at the boundary of the rUC and are given in Table 1 and illustrated in Fig. 5. Since in general,  $\mathbf{T} \neq \mathbf{I}$ , the load admissibility needs to be evaluated and  $\gamma$ , for each adjacent subdomain, determined. This is performed evaluating Eq. 25, and summarized in Table 2. Given a certain loading and using the data from Tables 1 and 2, Eq. 30 can be fully defined and periodic boundary conditions prescribed to the rUC.

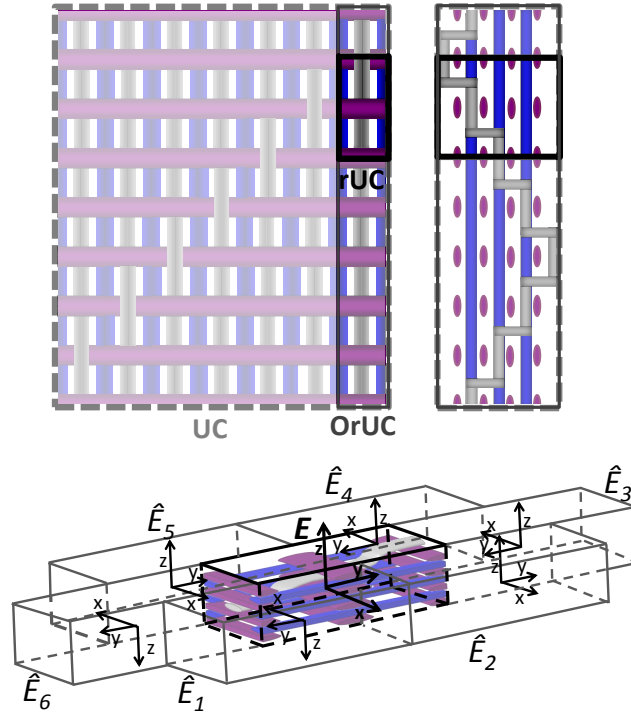


Figure 4: UC, OrUC and rUC of a 3D woven reinforcement architecture; representation of the reduced Unit Cell (rUC) and adjacent sub-domains.

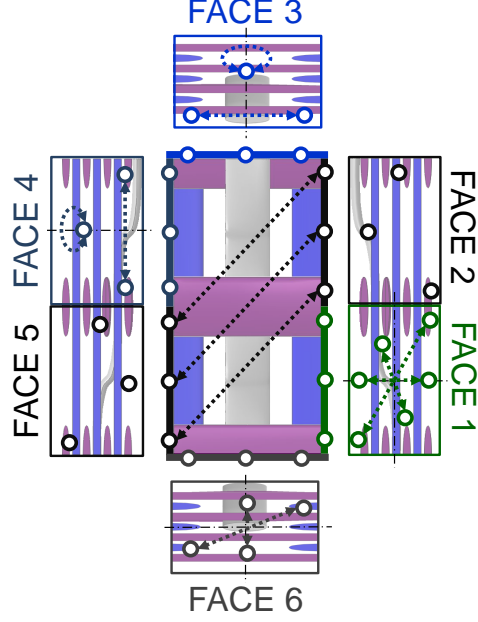


Figure 5: Geometrical relations between equivalent points at the boundary of the rUC.

Table 1: Geometrical relations between equivalent points at the boundary for the 3D woven rUC.  $l$ ,  $w$  and  $t$  are respectively, the length width and and thickness of the rUC.

	$\hat{E}_1$	$\hat{E}_2$	$\hat{E}_3$	$\hat{E}_4$	$\hat{E}_5$	$\hat{E}_6$
$\mathbf{T}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
$\mathbf{x}^{O\hat{E}}$	$\begin{bmatrix} w \\ -\frac{l}{2} \\ 0 \end{bmatrix}$	$\begin{bmatrix} w \\ \frac{l}{2} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ l \\ 0 \end{bmatrix}$	$\begin{bmatrix} -w \\ \frac{l}{2} \\ 0 \end{bmatrix}$	$\begin{bmatrix} -w \\ -\frac{l}{2} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -l \\ 0 \end{bmatrix}$
$\mathbf{x}^{\hat{A}}$	$\begin{bmatrix} x = \frac{w}{2} \\ -\frac{l}{2} \leq y \leq 0 \\ -\frac{t}{2} \leq z \leq \frac{t}{2} \end{bmatrix}$	$\begin{bmatrix} x = \frac{w}{2} \\ 0 \leq y \leq \frac{l}{2} \\ -\frac{t}{2} \leq z \leq \frac{t}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{w}{2} \leq x \leq \frac{w}{2} \\ y = \frac{l}{2} \\ -\frac{t}{2} \leq z \leq \frac{t}{2} \end{bmatrix}$	$\begin{bmatrix} x = -\frac{w}{2} \\ 0 \leq y \leq \frac{l}{2} \\ -\frac{t}{2} \leq z \leq \frac{t}{2} \end{bmatrix}$	$\begin{bmatrix} x = -\frac{w}{2} \\ -\frac{l}{2} \leq y \leq 0 \\ -\frac{t}{2} \leq z \leq \frac{t}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{w}{2} \leq x \leq \frac{w}{2} \\ y = -\frac{l}{2} \\ -\frac{t}{2} \leq z \leq \frac{t}{2} \end{bmatrix}$
$\mathbf{x}^{\hat{A}}$	$\begin{bmatrix} w - x_1^{\hat{A}} \\ -\frac{l}{2} - x_2^{\hat{A}} \\ -x_3^{\hat{A}} \end{bmatrix}$	$\begin{bmatrix} x_1^{\hat{A}} - w \\ x_2^{\hat{A}} - \frac{l}{2} \\ x_3^{\hat{A}} \end{bmatrix}$	$\begin{bmatrix} -x_1^{\hat{A}} \\ -x_2^{\hat{A}} + l \\ x_3^{\hat{A}} \end{bmatrix}$	$\begin{bmatrix} -x_1^{\hat{A}} - w \\ -x_2^{\hat{A}} + \frac{l}{2} \\ x_3^{\hat{A}} \end{bmatrix}$	$\begin{bmatrix} x_1^{\hat{A}} + w \\ x_2^{\hat{A}} + \frac{l}{2} \\ x_3^{\hat{A}} \end{bmatrix}$	$\begin{bmatrix} -x_1^{\hat{A}} \\ -x_2^{\hat{A}} - l \\ -x_3^{\hat{A}} \end{bmatrix}$

Table 2: Admissible loading cases and respective value of the load reversal factor  $\gamma_i$ , correspondent to each adjacent sub-domain  $\hat{E}_i$ , for the 3D woven rUC.

	$\gamma_i$	Admissible loading
Case 1	$[ 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 ]$	$\begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$
Case 2	$[ 1 \quad 1 \quad -1 \quad -1 \quad 1 \quad 1 ]$	$\begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{bmatrix}$

#### 4.2. Honeycombs

Honeycombs are other example of an extensively used periodic structure for which UC modelling and analysis can be simplified by the use of rUCs. Figure 6 shows the UC and rUC for a honeycomb structure. Following the procedure described previously, the geometrical relations between equivalent points at the boundary are first determined, Table 3, and Figs ?? and 7a. The load admissibility is evaluated and the load reversal factors  $\gamma_i$  found, Table 4.

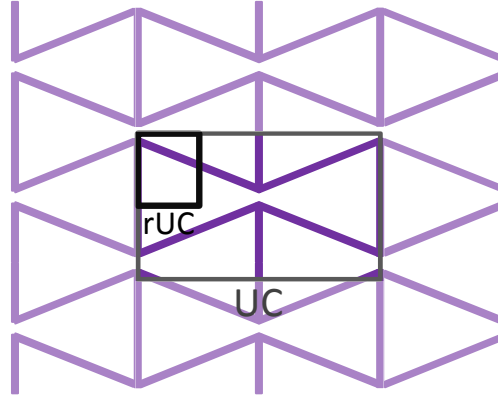


Figure 6: Unit cell (UC) and reduced Unit Cell (rUC) of a honeycomb structure.

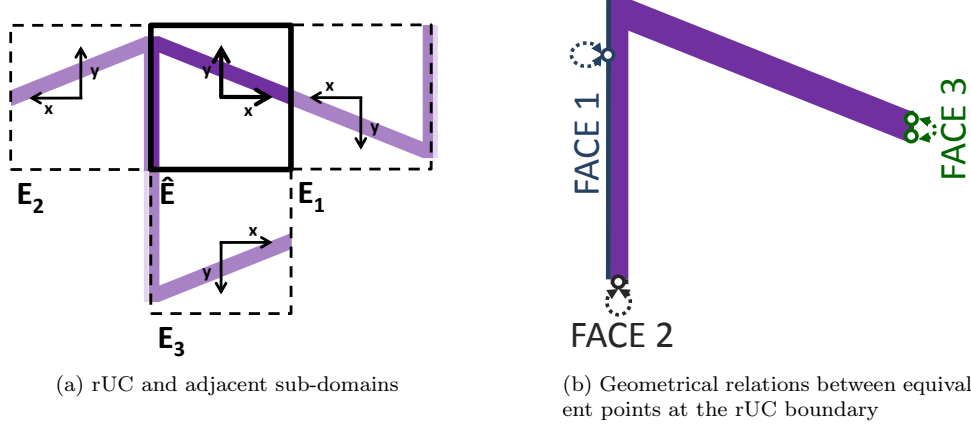


Figure 7:

Table 3: Geometrical relations between equivalent points at the boundary of the honeycomb rUC.  $l$ ,  $w$  are respectively, the length and width of the rUC.

	$\hat{E}_1$	$\hat{E}_2$	$\hat{E}_3$
$\mathbf{T}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
$x^{O\hat{E}}$	$\begin{bmatrix} w \\ 0 \end{bmatrix}$	$\begin{bmatrix} -w \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -l \end{bmatrix}$
$\mathbf{x}^{\hat{A}}$	$\begin{bmatrix} x = \frac{w}{2} \\ -\frac{l}{2} \leq y \leq \frac{l}{2} \end{bmatrix}$	$\begin{bmatrix} x = -\frac{w}{2} \\ -\frac{l}{2} \leq y \leq \frac{l}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{w}{2} \leq x \leq \frac{w}{2} \\ y = -\frac{l}{2} \end{bmatrix}$
$\mathbf{x}^A$	$\begin{bmatrix} -x_1^{\hat{A}} + w \\ -x_2^{\hat{A}} \end{bmatrix}$	$\begin{bmatrix} -x_1^{\hat{A}} - w \\ x_2^{\hat{A}} \end{bmatrix}$	$\begin{bmatrix} x_1^{\hat{A}} \\ -x_2^{\hat{A}} - l \end{bmatrix}$

Table 4: Admissible loading cases and respective value of the load reversal factor  $\gamma_i$ , correspondent to each adjacent sub-domain  $\hat{E}_i$ , for the rUC of the honeycomb.

	$\gamma_i$	Admissible loading
Case 1	$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix}$
Case 2	$\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & \sigma_{12} \\ \sigma_{21} & 0 \end{bmatrix}$



## 5. Conclusions

A theoretical framework leading to a sound derivation of PBCs for the analysis of domains smaller than the Unit Cells (UCs), named reduced Unit Cells (rUCs), by exploiting non-orthogonal translations and symmetries within the UC was developed. The investment in defining the problem formally resulted in a simple and readily usable formulation. The method is applied to two different periodic structures illustrating the potential of the rUC concept. Offset reduced Unit Cells are highlighted as a particular case with interesting features, allowing the analysis of domains smaller than the UC without any load restrictions.

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